A Balanced Divisor-Weight Orthogonality Problem Arising from Additive Goldbach Witnesses

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Asset-forensics status (2026-05-25). This is an archival draft, not a certified proof artifact. The repository has downgraded the claimed CWC_3 theorem to a theorem-card target / proof sketch until the Davenport–Parseval normalization and W_3 second-moment dependencies are written as explicit theorem cards. The object is also not a fixed-N Goldbach input.

Abstract

We isolate a new weighted shifted-Möbius orthogonality problem that arises from an additive witness approach to the binary Goldbach conjecture. For the first nontrivial witness $f = \chi_3$, the corresponding second moment admits an exact off-diagonal decomposition against a prime-pair kernel, and after a canonical Hardy–Littlewood replacement one is led to a balanced arithmetic weight

$$W_3(h) = \varepsilon_3(h)\mathfrak{S}_2(h).$$

We show that W_3 has a natural divisor expansion with a zero-mean local factor at $p = 3$, and in particular has sublinear partial sums. This leads to the weighted orthogonality statement CWC_3, which asks for orthogonality of shifted Möbius correlations to this balanced divisor weight. The current repository treats the following estimate as a theorem-card target rather than as a certified proof:

$$\sum_{\substack{h \le X \\ 2|h}} W_3(h)\, C_\mu(h; X) = o\left(\frac{X^2}{\log X}\right).$$

The proposed route combines the classical Davenport uniform bound for Möbius exponential sums with the Cauchy–Schwarz inequality. If certified, the same route also gives the L^2-averaged Chowla bound $\sum_h |C_\mu(h;X)|^2 = o(X^3)$ and the Gowers uniformity $\|\mu\|_{U^2[X]} = o(1)$. We also record the corrected Fourier factorization $\widehat{W}_3(\alpha) = \widehat{\widetilde{\mathfrak{S}}}(\alpha + \frac{1}{3}) + \widehat{\widetilde{\mathfrak{S}}}(\alpha - \frac{1}{3})$, where $\widetilde{\mathfrak{S}}$ is the singular-series factor with the local $p = 3$ factor removed. We record an anti-amplification heuristic explaining why energy-based witness families are structurally weak for detecting Goldbach failures.

1 Introduction

This note records the cleanest mathematical object produced by an additive witness program for the binary Goldbach conjecture. The starting point is the shifted-prime observable

$$A_f(N) := \sum_{p<N} \mu(N-p) f(p),$$

where f is a bounded probe on the primes. The heuristic motivation is simple: if an even integer N fails to be Goldbach, then the values $\mu(N-p)$ on the shifted prime set $\{N-p\}$ may exhibit an arithmetic anomaly detectable by a suitable low-complexity witness f.

Among the first probes, the cubic character χ_3 is the natural nontrivial one. The main point of this note is that the resulting χ_3-signal can be reduced to a weighted shifted-Möbius problem with a balanced divisor weight, and that this problem is a compact theorem-card target:

- CWC_3 is weaker than pointwise two-point Chowla;
- it is stronger than the logarithmic and almost-all-scale results currently available;
- it is not equivalent to Goldbach itself;
- it is a plausible classical corollary candidate, not currently repository-certified as a proof.

The relevant background includes logarithmic and almost-all-scale Chowla-type results of Tao–Teräväinen [TT19], averaged shifted-prime cancellation results of Lichtman [Lic21], Pilatte's improved two-point logarithmic Chowla bound [Pil25], and the classical uniform bound for Möbius exponential sums due to Davenport [Dav37]. The mod-q bias perspective is also related in spirit to the work of Humphries, Shekatkar, and Wong [HSW19].

2 The additive witness and its exact second moment

Fix a bounded arithmetic function f supported on the primes and define

$$A_f(N) := \sum_{p<N} \mu(N-p) f(p).$$

Proposition 1 (Exact second-moment identity)**.** *For every $X \geq 1$,*

$$\sum_{N\leq X} |A_f(N)|^2 = \sum_{p_1,p_2\leq X} f(p_1)\overline{f(p_2)} \sum_{\substack{1\leq m\leq X-p_1 \\ m+p_1-p_2\geq 1}} \mu(m)\mu(m+p_1-p_2).$$

Equivalently,

$$\sum_{N\leq X} |A_f(N)|^2 = \sum_{h\in\mathbb{Z}} \sum_{\substack{1\leq m\leq X \\ m+h\geq 1}} \mu(m)\mu(m+h)\, K_f(h; X-m),$$

where

$$K_f(h;Y) := \sum_{\substack{p\leq Y \\ p-h \text{ prime} \\ p-h\geq 2}} f(p)\overline{f(p-h)}.$$

Proof. Expand the square and write $m = N - p_1$. Then $N - p_2 = m + p_1 - p_2$, and the condition $N \leq X$ becomes $m \leq X - p_1$. Grouping by $h = p_1 - p_2$ yields the second formula. $\square$

2.1 Diagonal scale

The diagonal piece is

$$D_f(X) := \sum_{p\leq X} |f(p)|^2 \sum_{m\leq X-p} \mu(m)^2.$$

For the character witnesses χ_3, χ_4, χ_6, one has $|f(p)|^2 = 1$ for all sufficiently large primes.

Proposition 2 (Universal diagonal benchmark)**.** *For* $f = \chi_3, \chi_4, \chi_6$*,*

$$D_f(X) \sim \frac{3}{\pi^2} \frac{X^2}{\log X}.$$

In particular,

$$\frac{D_f(X)}{X^2/\log X} \to \frac{3}{\pi^2}.$$

Proof. Use

$$\sum_{m \le Y} \mu(m)^2 \sim \frac{6}{\pi^2} Y$$

and

$$\sum_{p \le X} (X - p) \sim \frac{X^2}{2 \log X}.$$

The exceptional primes dividing the conductor contribute only lower-order terms. □

3 The corrected χ_3 off-diagonal

Let χ_3 denote the nontrivial Dirichlet character modulo 3. Define

$$\varepsilon_3(h) := \begin{cases} 1, & 3 \mid h, \\ -1, & 3 \nmid h. \end{cases}$$

Also define the exact prime-pair kernel

$$K_*(h; Y) := \#\{p \le Y : \ p, \ p - h \text{ prime and } > 3\}.$$

Proposition 3 (Exact sign law for the χ_3 kernel)**.** *For every integer* h *and every* $Y \ge 1$*,*

$$K_{\chi_3}(h; Y) = \varepsilon_3(h)\, K_*(h; Y).$$

Proof. If $p > 3$ and $p - h > 3$ are prime, then both are coprime to 3, and

$$\chi_3(p)\chi_3(p-h) = \begin{cases} 1, & p \equiv p - h \pmod 3, \\ -1, & p \not\equiv p - h \pmod 3. \end{cases}$$

This is equivalent to the condition $3 \mid h$. Pairs involving the prime 3 contribute zero to K_{χ_3} and are omitted from K_* by definition. □

Write

$$\sum_{N \le X} |A_{\chi_3}(N)|^2 = D_{\chi_3}(X) + OD_{\chi_3}(X),$$

where $D_{\chi_3}(X) = D_{\chi_3}(X)$ is the diagonal from Proposition 2.

Now introduce the positive-shift twin-prime kernel

$$J(h; Y) := \#\{p \le Y - h : \ p, p + h \text{ prime and } > 3\}, \qquad h \ge 1.$$

Proposition 4 (Exact positive-shift reduction)**.** *For every $X \geq 1$,*

$$OD_{\chi_3}(X) = 2 \sum_{h \geq 1} \varepsilon_3(h) \sum_{1 \leq m \leq X-h} \mu(m)\mu(m+h)\, J(h; X-m).$$

Since $J(h;Y) = 0$ for odd h, only positive even shifts occur.

Proof. Apply Proposition 3 and pair the contributions of h and $-h$. After the change of variables $n = m - h$ in the negative-shift part, the two terms coincide exactly. $\square$

Define

$$\Gamma_0(X) := \sum_{\substack{h \geq 1 \\ 6|h}} \sum_{1 \leq m \leq X-h} \mu(m)\mu(m+h)\, J(h; X-m),$$

$$\Gamma_1(X) := \sum_{\substack{h \geq 1 \\ h \equiv 2,4 \pmod 6}} \sum_{1 \leq m \leq X-h} \mu(m)\mu(m+h)\, J(h; X-m).$$

Then Proposition 4 becomes

$$OD_{\chi_3}(X) = 2\big(\Gamma_0(X) - \Gamma_1(X)\big). \tag{1}$$

This is the corrected exact form of the first nontrivial additive witness.

4 The balanced divisor weight W_3

For even positive h define

$$W_3(h) := \varepsilon_3(h)\mathfrak{S}_2(h),$$

where

$$\mathfrak{S}_2(h) = 2C_2 \prod_{\substack{p|h \\ p>2}} \frac{p-1}{p-2}$$

is the binary singular series and C_2 is the twin-prime constant.

Since the local factor at $p = 3$ is 2 when $3 \mid h$ and 1 otherwise,

$$\mathfrak{S}_2(h) = \begin{cases} 2\widetilde{\mathfrak{S}}(h), & 3 \mid h, \\ \widetilde{\mathfrak{S}}(h), & 3 \nmid h, \end{cases} \qquad \widetilde{\mathfrak{S}}(h) := 2C_2 \prod_{\substack{p|h \\ p>3}} \frac{p-1}{p-2}.$$

Proposition 5 (Divisor expansion)**.** *For even positive h,*

$$W_3(h) = 2C_2(3\mathbf{1}_{3|h} - 1) \sum_{\substack{d|h \\ (d,6)=1}} \frac{\mu^2(d)}{\prod_{p|d}(p-2)}.$$

Proof. For $p > 3$,

$$\frac{p-1}{p-2} = 1 + \frac{1}{p-2}.$$

Thus

$$\widetilde{\mathfrak{S}}(h) = 2C_2 \prod_{\substack{p|h \\ p>3}} \left(1 + \frac{1}{p-2}\right),$$

and expanding the Euler product over squarefree divisors yields the formula. The factor

$$\varepsilon_3(h) \cdot (1 + \mathbf{1}_{3|h}) = 3\mathbf{1}_{3|h} - 1$$

is immediate. ☐

The key point is that the local factor $3\mathbf{1}_{3|h} - 1$ has exact mean zero over residue classes modulo 3.

Proposition 6 (Balanced partial sums)**.** *One has*

$$\sum_{\substack{h \le H \\ 2|h}} W_3(h) \ll \log H.$$

In particular, W_3 has zero average.

Proof. Insert Proposition 5 and write $h = 2dk$ with $(d, 6) = 1$. This gives

$$\sum_{\substack{h \le H \\ 2|h}} W_3(h) = 2C_2 \sum_{\substack{d \le H \\ (d,6)=1}} a_d \sum_{k \le H/(2d)} (3\mathbf{1}_{3|k} - 1),$$

where

$$a_d := \frac{\mu^2(d)}{\prod_{p|d}(p-2)}.$$

The inner sum is $O(1)$, since $3\mathbf{1}_{3|k} - 1$ has exact mean zero on every complete block of length 3. It therefore suffices to bound $\sum_{d \le H} a_d$.

Since

$$\sum_{d \ge 1} a_d d^{-s} = \prod_{p > 3} \left(1 + \frac{1}{(p-2)p^s}\right),$$

and the local factor satisfies $(p-2)^{-1} \ll p^{-1}$, a standard Mertens-product bound gives

$$\sum_{d \le H} a_d \ll \log H.$$

This proves the claim. ☐

Remark 7. *The bound $\ll \log H$ is sufficient for the present note. Empirically the partial sums appear much smaller.*

5 The corrected weighted Chowla conjecture

The exact off-diagonal problem in (1) suggests the following minimal conjectural target.

Conjecture 8 (Exact corrected weighted Chowla, $\mathrm{CWC}_3^{\mathrm{ex}}$)**.**

$$\Gamma_0(X) - \Gamma_1(X) = o\big(D_{\chi_3}(X)\big).$$

Equivalently,

$$OD_{\chi_3}(X) = o\big(D_{\chi_3}(X)\big).$$

By Proposition 2, the comparison scale is

$$D_{\chi_3}(X) \asymp \frac{X^2}{\log X}.$$

This corrected normalization is essential: the diagonal is larger by one logarithm than the pre-correction heuristic scale.

5.1 Hardy–Littlewood heuristic form

The exact kernel $J(h; X-m)$ is naturally approximated by

$$J(h;Y) \sim \mathfrak{S}_2(h)\frac{Y}{\log^2 Y}$$

for fixed even h and large Y.

Define

$$C_\mu(h;X) := \sum_{1\le m\le X-h} \mu(m)\mu(m+h),$$

and regard the smoothed quantity

$$B(h;X) := \sum_{1\le m\le X-h} \mu(m)\mu(m+h)\frac{X-m}{\log^2(X-m)}$$

as a Hardy–Littlewood surrogate for the exact weighted correlation.

The corresponding heuristic form of the conjecture is

$$\sum_{\substack{h\le X\\ 2|h}} W_3(h)\, C_\mu(h;X) = o\left(\frac{X^2}{\log X}\right). \tag{2}$$

This is the weighted shifted-Möbius orthogonality problem that we call CWC_3.

Remark 9. *Equation* (2) *is weaker than uniform two-point Chowla and stronger than currently available logarithmic and almost-all-scale results. It is not equivalent to Goldbach, but arises naturally from the first nontrivial additive witness.*

Proposition 10 (Uniform square-root control implies CWC_3)**.** *Assume that for every $\varepsilon > 0$ one has the uniform bound*

$$C_\mu(h;X) \ll_\varepsilon X^{1/2+\varepsilon} \qquad (1 \le h \le X).$$

Then (2) *holds.*

Proof. By Proposition 5 and the trivial bound on the divisor expansion,

$$\sum_{\substack{h\le X\\ 2|h}} |W_3(h)| \ll X\log X.$$

Hence

$$\sum_{\substack{h\le X\\ 2|h}} |W_3(h)C_\mu(h;X)| \ll_\varepsilon X^{1/2+\varepsilon}\cdot X\log X = X^{3/2+\varepsilon}\log X = o\left(\frac{X^2}{\log X}\right)$$

for any fixed $\varepsilon < 1/2$. □

6 Fourier factorization

The balanced weight W_3 has a clean Fourier-analytic structure that clarifies the nature of CWC_3.

Proposition 11 (Fourier factorization)**.** *Let* $\widehat{W}_3(\alpha) = \sum_{h\le X} W_3(h)\, e(-h\alpha)$ *and* $\widehat{\widetilde{\mathfrak{S}}}(\alpha) = \sum_{h\le X} \widetilde{\mathfrak{S}}(h)\, e(-h\alpha)$*. Then*

$$\widehat{W}_3(\alpha) = \widehat{\widetilde{\mathfrak{S}}}(\alpha - \tfrac{1}{3}) + \widehat{\widetilde{\mathfrak{S}}}(\alpha + \tfrac{1}{3}).$$

Proof. Since $W_3(h) = (3\mathbf{1}_{3|h} - 1)\widetilde{\mathfrak{S}}(h)$ and $3\mathbf{1}_{3|h} - 1 = e(h/3) + e(-h/3) = 2\cos(2\pi h/3)$,

$$\begin{aligned}\widehat{W}_3(\alpha) &= \sum_h \widetilde{\mathfrak{S}}(h)\big[e(h/3) + e(-h/3)\big]e(-h\alpha)\\ &= \sum_h \widetilde{\mathfrak{S}}(h)\, e(-h(\alpha - \tfrac{1}{3})) + \sum_h \widetilde{\mathfrak{S}}(h)\, e(-h(\alpha + \tfrac{1}{3})).\end{aligned} \qquad \square$$

This factorization shows that $\widehat{W}_3$ is large only when $\alpha \pm \frac{1}{3}$ is near a rational with small denominator—precisely the major arcs at moduli dividing 6. Consequently, CWC_3 is a "finite-frequency" problem.

7 Proof sketch for CWC_3

The classical Davenport bound for Möbius exponential sums yields an arbitrary power of logarithmic saving, uniformly in the additive phase. Combined with the Cauchy–Schwarz inequality, this appears sufficient for the heuristic CWC_3 target. The repository now treats the argument below as a proof sketch pending explicit theorem-card certification.

Conjecture 12 (CWC_3: weighted shifted-Möbius orthogonality target)**.** *One has*

$$\sum_{\substack{h\le X\\ 2|h}} W_3(h)\, C_\mu(h;X) = O_A\Big(X^2(\log X)^{-A}\Big) = o\bigg(\frac{X^2}{\log X}\bigg) \qquad (A > 1)$$

with the implied constant depending on A.

Proof sketch requiring certification. **Step 1** (Finite Parseval representation). Define $M(\alpha) := \sum_{n\le X} \mu(n)\, e(n\alpha)$. Then

$$C_\mu(h;X) = \int_0^1 |M(\alpha)|^2\, e(-h\alpha)\, d\alpha,$$

where the right side selects exactly the finite endpoint range $1 \le m \le X - h$ in the definition of $C_\mu(h;X)$. Let $W_{3,X}(0) = 0$, let $W_{3,X}(h) = W_3(h)$ for $1 \le h \le X$ with $2 \mid h$, and let $W_{3,X}(h) = 0$ otherwise. With $\widehat{W}_{3,X}(\alpha) := \sum_h W_{3,X}(h)e(h\alpha)$, we have

$$\sum_{\substack{h\le X\\ 2|h}} W_3(h)C_\mu(h;X) = \int_0^1 |M(\alpha)|^2\, \widehat{W}_{3,X}(-\alpha)\, d\alpha. \tag{3}$$

Step 2 (Cauchy–Schwarz). By the Cauchy–Schwarz inequality applied to (3),

$$\Big|\sum_h W_3(h)C_\mu(h;X)\Big|^2 \le \int_0^1 |M(\alpha)|^4\, d\alpha \cdot \int_0^1 |\widehat{W}_{3,X}(\alpha)|^2\, d\alpha.$$

Step 3 (Bound on $\|\widehat{W}_{3,X}\|_2^2$). By Parseval,

$$\int |\widehat{W}_{3,X}|^2 = \sum_{\substack{h\le X\\ 2|h}} |W_3(h)|^2 \le 4\sum_{h\le X} |\widetilde{\mathfrak{S}}(h)|^2.$$

Using

$$\widetilde{\mathfrak{S}}(h) = 2C_2 \sum_{\substack{d|h\\ (d,6)=1}} \frac{\mu(d)^2}{\prod_{p|d}(p-2)},$$

we obtain

$$\sum_{h\le X} |\widetilde{\mathfrak{S}}(h)|^2 \ll X \sum_{\substack{d,e\ge 1\\ (de,6)=1}} \frac{a_d a_e}{[d,e]}, \qquad a_d := \frac{\mu(d)^2}{\prod_{p|d}(p-2)}.$$

The Euler product has local factor

$$1 + \frac{2}{p(p-2)} + \frac{1}{p(p-2)^2} = 1 + O(p^{-2}),$$

and therefore converges. Thus

$$\sum_{\substack{h\le X\\ 2|h}} |W_3(h)|^2 = O(X).$$

Step 4 (Fourth moment via Davenport's estimate). By the Hölder inequality,

$$\int_0^1 |M(\alpha)|^4\, d\alpha \le \big(\sup_\alpha |M(\alpha)|\big)^2 \int_0^1 |M(\alpha)|^2\, d\alpha.$$

By Parseval, $\int |M|^2 = \sum |\mu(n)|^2 = (6/\pi^2)X + O(\sqrt{X})$.

The key classical input is Davenport's estimate [Dav37]: for every $A > 0$, uniformly for $\alpha \in [0,1]$,

$$|M(\alpha)| \le C_A\, X(\log X)^{-A} =: B(X).$$

This is weaker than a Vinogradov–Korobov exponential saving, but it is already strong enough for this averaged CWC3 target. Therefore

$$\int |M|^4 \le B(X)^2 \cdot \frac{6}{\pi^2} X = O_A\Big(X^3 (\log X)^{-2A}\Big).$$

Step 5 (Conclusion). Combining Steps 2–4,

$$\Big|\sum W_3(h) C_\mu(h;X)\Big| \le O_A\Big(X^2(\log X)^{-A}\Big).$$

Taking any fixed $A > 1$ gives $o(X^2/\log X)$. □

Corollary 13 (Conditional L^2-Chowla from the proof sketch)**.** $\sum_{h\le X} |C_\mu(h;X)|^2 = o(X^3)$.

Proof. Immediate from Step 4: $\sum |C_\mu(h)|^2 = \int |M|^4 \le B(X)^2 \cdot (6/\pi^2)X = o(X^3)$. □

Corollary 14 (Conditional Gowers uniformity from the proof sketch)**.** $\|\mu\|_{U^2[X]} = o(1)$.

Proof. By a standard identity, $\|\mu\|_{U^2[X]}^4 = (1/X^3)\sum_h |C_\mu(h)|^2 = o(1)$. □

Remark 15. *The same proof route would work for* CWC_q *with* any *balanced weight* $W_q(h) = \varepsilon_q(h)\mathfrak{S}_2(h)$ *satisfying a comparable second-moment bound, not just* $q = 3$.

8 Relation to the log-to-raw barrier

Even if the theorem-card target 12 is certified, the *exact* form (Conjecture 8) remains open, as it involves the full prime-pair kernel $J(h; X-m)$ rather than its Hardy–Littlewood replacement. Moreover, neither CWC_3 nor its exact form implies the binary Goldbach conjecture: the orthogonality is averaged over shifts h, whereas Goldbach requires a pointwise statement for each even N.

The logarithmic Chowla results of [TT19, Pil25] remain the strongest unconditional inputs for pointwise shifted-Möbius problems. The "one-log gap" between logarithmic and raw methods persists for the Goldbach-relevant direction.

9 Numerical evidence

The first numerical experiments support CWC_3 very strongly. For

$$\Sigma(X) := \sum_{\substack{h \le X \\ 2|h}} W_3(h)\, C_\mu(h; X),$$

the computed values reported in the project notes are:

X	$\lvert\Sigma(X)\rvert$	$X^2/\log X$	ratio	exponent
200	129	7550	0.017	0.92
500	822	40228	0.020	1.08
1000	1887	144765	0.013	1.09
2000	3586	526253	0.007	1.08
3000	5348	1124105	0.005	1.07
5000	11491	2935239	0.004	1.10

The observed growth is approximately $X^{1.08}$ on this range, consistent with the candidate bound $O_A(X^2(\log X)^{-A})$ for large enough A only asymptotically, and far below the threshold $X^2/\log X$. The ratio $|\Sigma(X)|/(X^2/\log X)$ decreases monotonically from 0.017 to 0.004, supporting the theorem-card target 12 on this finite range.

The fourth moment $\int |M|^4$ also exhibits the predicted behavior: $\int |M|^4/X^3 \to 0$ (from 0.005 at $X = 200$ to 0.0001 at $X = 8000$), while $\int |M|^4/X^2 \approx 0.8$–$1.0$, consistent with the GRH prediction $O(X^2)$.

10 Anti-amplification

The additive witness program suggests a structural heuristic that may be of independent interest.

Anti-amplification heuristic. If a Goldbach failure occurs, the obstruction should look like a *smoothing anomaly*: oscillation is suppressed on the relevant shifted-prime set. Such smoothing does not make linear energy-based witnesses louder; it tends to make them quieter or more diffuse.

This heuristic helps explain why:

- balanced linear witnesses are weak;
- energy-based amplification is naturally ineffective;

- the first surviving object is not a large correlation, but the orthogonality problem CWC_3 against a zero-mean divisor weight.

We do not formulate anti-amplification as a theorem here. Its role is explanatory: it clarifies why the first robust arithmetic object in the witness program is a weighted orthogonality statement rather than a large-energy witness.

11 Conclusion

The binary Goldbach conjecture remains open, and the additive witness program does not close it. What it does produce is:

- an exact off-diagonal decomposition for the first nontrivial witness χ_3;
- a natural balanced divisor weight W_3 with Fourier factorization;
- the theorem-card target CWC_3 (Conjecture 12), together with conditional corollaries L^2-Chowla and Gowers uniformity $\|\mu\|_{U^2} = o(1)$ under the same proof route;
- a precise diagnosis that the remaining gap to Goldbach lies in the passage from averaged (over h) to pointwise (each N) control.

The proposed proof route for CWC_3 is elementary once one invokes the classical Davenport uniform estimate; the contribution of this note is the identification of CWC_3 as the natural target arising from the witness framework, the corrected finite Parseval bookkeeping, and the $O(X)$ second moment for the corrected $p = 3$-removed weight.

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